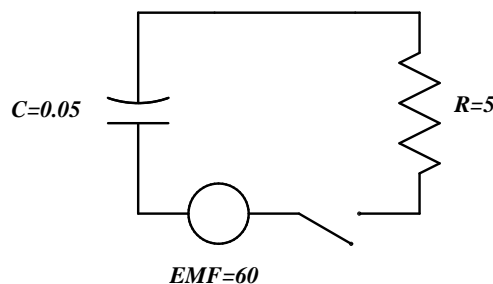## III. Example 2: R-C DC Circuit

Questions:

Physical characteristics of the circuit: 60 volt DC battery connected in series with a 0.05 farad capacitor and a 5 ohm resistor. There is no charge on the capacitor and current flows when the open switch is closed. (Note: This is Exercise #27 on p. 521 and p. 528 of Stewart: **Calculus—Concepts and Contexts**, 2nd ed.)

Task: Write down the Initial Value Problem associated with this circuit and solve it for the charge in order to answer the following questions.

[a] Describe in words how the charge changes over time.
[b] What is the charge 0.5 second after the switch is closed?
[c] At what time does the charge equal 90% of the steady-state charge?
[d] What is the average charge over the first five time units for this circuit?

Solution: By Kirchhoff's laws we have $E_R + E_C = EMF$ which, with $E_R = R \cdot Q'(t)$ and $E_C = Q(t)/C$, translates into the following Initial Value Problem (for $t \geq 0$):

$$5\,Q'(t) + \frac{Q(t)}{0.05} = 60, \qquad Q(t) = 0 \quad \text{at} \quad t = 0$$

We can solve for Q using the method of separation of variables.

Outline of solution by *separation of variables*

First, we will divide the ODE through by 5, replace $Q(t)$ by Q, and use the differential notation for derivatives:

$$\frac{dQ}{dt} + 4\,Q = 12$$

Next, use algebra to rewrite this as

$$\frac{dQ}{12 - 4\,Q} = dt$$

and integrate both sides to obtain

$$-\frac{1}{4}\ln|12 - 4\,Q| = t + C$$

which with the initial condition $Q(0) = 0$ yields the circuit charge

$$Q(t) = 3 - 3\,e^{-4t}, \quad t \geq 0$$

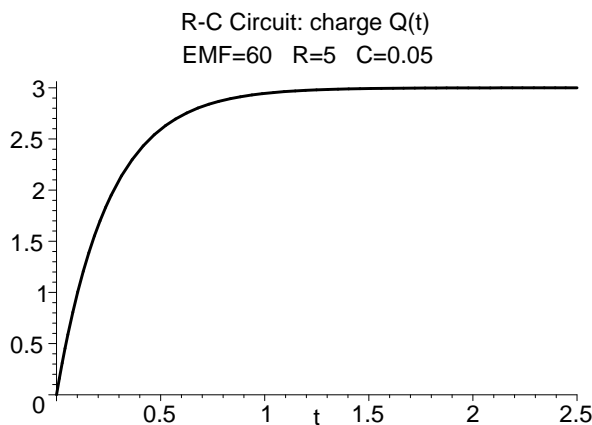More details for all these steps may be found below, after the Answers.

Answers:

[a] Describe in words how the charge changes over time.

The following graph shows how $Q(t)$ increases from 0 at $t = 0$ toward an asymptotic limit 3 as t increases:

$$\lim_{t \to \infty} Q(t) = 3 - 3 \lim_{t \to \infty} e^{-4t} = 3 - 3(0) = 3$$

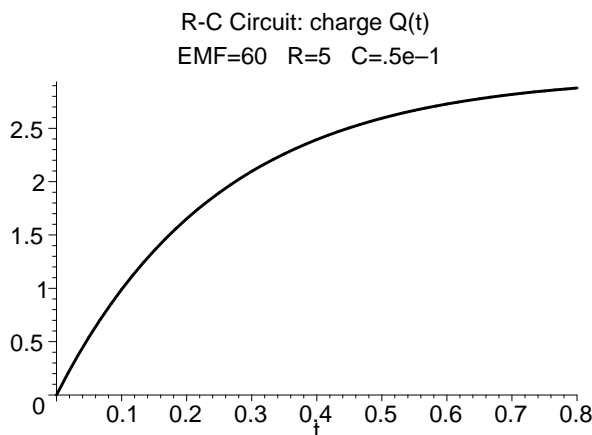This asymptotic limit is called the *steady-state* charge.

R-C Circuit: charge Q(t)
EMF=60   R=5   C=0.05

[b] What is the charge 0.5 second after the switch is closed?

$$Q(0.5) = 3 - 3\,e^{-4(0.5)} = 3 - 3\,e^{-2} \approx 2.59$$

which looks correct according to the following graph of $Q(t)$.

R-C Circuit: charge Q(t)
EMF=60   R=5   C=.5e–1

[c] At what time does the charge equal 90% of the steady-state charge?

Solve

$$Q(t) = 0.90(3)$$

or

$$3 - 3\,e^{-4\cdot t} = 2.7$$

to get

$$t = -\frac{1}{4}\ln(0.1) \approx 0.576$$

This answer could have been approximated by graphing $Q(t)$ on your calculator and zooming or tracing the curve. The preceding graph of $Q(t)$ provides a visual check of the answer.

[d] What is the average charge over the first five time units for this circuit?

By definition, the time unit for this R-C DC circuit is

$$\tau = C \cdot R = 0.05 \cdot 5 = 0.25$$

By definition of the *average value of a function* (see, e.g., p. 473 of Stewart), the average charge over first five time units is

$$Q_{avg} = \frac{1}{1.25} \int_0^{1.25} \left(3 - 3\,e^{-4t}\right) dt = \frac{3}{5}\left(4 + e^{-5}\right) \approx 2.40 \text{ coulombs}$$

Details of solution by *separation of variables*

After multiplying both sides of the ODE

$$\frac{dQ}{dt} = 12 - 4\,Q$$

by dt, we get the ODE in differential form

$$dQ = (12 - 4\,Q)\,dt$$

Divide both sides by $12 - 4\,Q$ in order to *separate variables: put anything involving Q on one side and anything involving t on the other side*:

$$\frac{dQ}{12 - 4\,Q} = dt \tag{1}$$

Now we are allowed to integrate each side separately and still have equality. The right side of equation (1) is easy:

$$\int dt = t + C$$

where C is an arbitrary constant. The left side of equation (1) looks related to the integral $\int \frac{1}{x}\,dx$. So we use the substitution

$$x = 12 - 4\,Q$$

$$\text{to get} \qquad \frac{dx}{dQ} = -4$$

$$\text{or} \qquad dQ = -\frac{1}{4}dx$$

Then in equation (1) we replace $12 - 4\,Q$ with x and dQ with $-\frac{1}{4}dx$ and integrate in order to get the left side to equal

$$\int \frac{1}{12 - 4\,Q}\,dQ = \int \frac{1}{x}\left(-\frac{1}{4}dx\right)$$

$$= -\frac{1}{4}\int \frac{1}{x}\,dx$$

$$= -\frac{1}{4}\ln|x| + C$$

$$= -\frac{1}{4}\ln|12 - 4\,Q| + C$$

Hence equation (1), after both sides are integrated, becomes (collecting all arbitrary constants on the right hand side as a single arbitrary constant)

$$-\frac{1}{4}\ln|12 - 4\,Q| = t + C \tag{2}$$

Since there is no charge when the switch is thrown, we let $Q = 0$ when $t = 0$ to solve for C

$$-\frac{1}{4}\ln|12 - 0| = 0 + C \Longrightarrow C = -\frac{1}{4}\ln 12$$

and so equation (2) becomes

$$-\frac{1}{4}\ln|12 - 4\,Q| = -\frac{1}{4}\ln 12 + t$$

It is usually preferable to solve for the dependent variable, Q in this case. To do that, we first multiply both sides of the last equation by -4 to get

$$\ln|12 - 4\,Q| = \ln 12 - 4t$$

then take the exponential (inverse logarithm) of both sides

$$e^{\ln|12 - 4\,Q|} = e^{\ln 12 - 4t} \tag{3}$$

and then use a property of exponentials

$$e^{a+b} = e^a \times e^b$$

with $a = \ln 12$ and $b = -4t$ to get from equation (3)

$$\begin{aligned}|12 - 4\,Q| &= e^{\ln 12} \times e^{-4t} \\ &= 12\,e^{-4t}\end{aligned}$$

since $e^{\ln 12} = 12$. Now $|x| = c \Longrightarrow x = \pm c$ so we have

$$12 - 4\,Q = \pm 12\,e^{-4t}$$

Since we know that $Q = 0$ at $t = 0$, we determine the sign to be $+$, allowing us to solve for Q by dividing both sides of the last equation by 4 and then isolating Q on one side

$$Q(t) = 3\left(1 - e^{-4t}\right), \quad t \geq 0$$